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What is the gravity dual of a chiral primary?

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Abstract

In the AdS/CFT correspondence a chiral primary is described by a supergravity solution with mass equaling angular momentum. For $AdS_3 \times S^3$ we are led to consider three special families of metrics with this property: metrics with conical defects, Aichelburg-Sexl type metrics generated by rotating particles, and metrics generated by giant gravitons. We find that the first two of these are special cases of the complete family of chiral primary metrics which can be written down using the general solution in hep-th/0109154, but they correspond to two extreme limits – the conical defect metrics map to CFT states generated by twist operators that are all identical, while the Aichelburg-Sexl metrics yield a wide dispersion in the orders of these twists. The giant graviton solutions in contrast do not represent configurations of the D1-D5 bound state; they correspond to fragmenting this system into two or more pieces. We look at the large distance behavior of the supergravity fields and observe that the excitation of these fields is linked to the existence of *dispersion* in the orders and spins of the twist operators creating the chiral primary in the CFT.

1 Introduction

1.1 The issue

Consider the AdS/CFT duality map when the spacetime is $AdS_3 \times S^3 \times M_4$. The dual theory is a 1+1 dimensional CFT obtained from the low energy limit of the D1-D5 bound state [1].

More specifically, consider 'global AdS_3' $\times S^3 \times M_4$. The state of the CFT dual to this geometry is the Neveu-Schwarz (NS) vaccum $|0\rangle_{NS}$. This vacuum is the simplest state of the CFT, with $h = j = \bar{h} = \bar{j} = 0$. The next simplest states are expected to be the chiral primaries of the CFT, with $h = j, \bar{h} = \bar{j}$. What are the metrics dual to these states of the CFT?

By the general nature of the AdS/CFT map we expect that the dual of a chiral primaries will be created by putting an appropriate massless particle from the supergravity multiplet into the geometry; the particle rotates on S^3 with angular momentum $(j, \bar{j})$ and sits at the origin of AdS_3 (so that we get 'minimum energy for given charge'). In [2] the wavefunction for such a quantum was computed – it was given by an appropriate spherical harmonic on S^3 and was localized around $r = 0$ in AdS_3.

In the CFT, we can list chiral primaries most easily at the 'orbifold point'. At this point in moduli space the CFT is described by a $N = 4$ supersymmetric sigma model with target space $(M_4)^N/S_N$, the symmetric product of $N = n_1 n_5$ copies of M_4 [3]. The state dual to the supergravity quantum mentioned above is given by a 'twist' operator acting on the vacuum[1]

$$\sigma_n^{--}|0\rangle_{NS} \tag{1.1}$$

The operator σ_n^{--} has $h = j = \bar{h} = \bar{j} = \frac{n-1}{2}$.

We are interested in studying the metrics dual to the chiral primaries. Thus we want the backreaction of the supergravity quanta on the geometry. This backreaction will be described classically when we consider a large number of massless quanta orbiting the S^3. In the CFT we expect that the corresponding metric will be dual to a chiral primary

$$\sigma_{n_1}^{--}\sigma_{n_2}^{--}\ldots\sigma_{n_k}^{--}|0\rangle_{NS} \tag{1.2}$$

Let us see what we expect to be the nature of the corresponding geometry. A priori, we are led to consider three possibilities:

(a) *Conical defect metrics:* We can regard the supergravity theory as a 2+1 dimensional gravity theory arising from dimensional reduction on $S^3 \times M_4$. A particle with angular momentum on S^3 appears as a particle with mass $m =$ in 2+1 dimensions. But in 2+1 gravity massive particles create 'conical defect metrics' so we may expect that the

[1]The twist operator σ_n creates a cyclic permutation of n copies of M_4 around its insertion point. The superscripts $\{--\}$ indicate that this chiral primary corresponds to the $(0, 0)$ form on M_4; action of fermion zero modes can raise the indices upto $\{++\}$ which corresponds to the top form $(2, 2)$ on M_4. For a detailed study of these chiral primaries see [4, 5].

metrics dual to chiral primaries (1.2) will have such a conical defect structure in some way.

(b) *Aichelburg-Sexl metric:* Now consider the particles from the viewpoint of 5+1 dimensional $AdS_3 \times S^3$ spacetime. A chiral primary state has the least energy for given angular momentum. At least naively, this suggests that we must have all the massless particles rotating on a common diameter of the S^3, at the origin in AdS_3. If we go close to this diameter, we just see a uniform line of massless particles, moving at the speed of light along this line, in a spacetime that is essentially $5+1$ dimensional flat spacetime in the neighborhood of the line. The metric of a massless particle in flat space is the Aichelburg-Sexl metric, which is readily extended to a line of massless particles. We thus expect the metric to be this Aichelburg-Sexl type near the diameter, while going over to $AdS_3 \times S^3$ at infinity.

But such a metric does *not* have a conical defect structure; metrics in dimensions higher than 2+1 have power law behavior near the singularity. Thus if such a solution were the correct one then it would certainly be different from (a) above.

(c) *Giant gravitons:* There is yet a third possibility for the metric that we need to consider. It was shown in [6] that massless quanta rotating on the sphere with high angular momenta can expand to large radius objects while maintaining the condition $\Delta = J$. In other examples of $AdS_n \times S^m$ spacetimes the radius of the 'giant graviton' increases with its angular momentum J, but for $AdS_3 \times S^3$ we have a slightly different situation. Giant gravitons can exist only at certain points in the moduli space. Such a special point is obtained for example in the case when $M_4 = T^4$ when all the gauge fields are set to zero. At these special points a graviton can become 'giant' whenever its angular momentum J is a multiple of n_5. At such J the potential for the radial size of the graviton is 'flat' so the graviton can have any diameter while maintaining $\Delta = J$.

Geometries with a classical value for total angular momentum will have $J \gg n_5$, so we can obtain the desired angular momentum by using a large number of giant gravitons. But since these objects are extended, we expect to get a metric that differs from both the metrics suggested in (a) and (b) above.

We can thus restate our basic question as: Which of these three kinds of metrics correspond to chiral primaries of the CFT?

It is possible to construct explicitly metrics of *each* of the above three types. Each metric is a classical solution of the full supergravity equations, and has energy and angular momentum which translate to $\Delta = J$. This sharpens the above question – what is the correct dual of the chiral primaries of the CFT?

1.2 Results

In short, what we find is the following. We can construct metrics dual to *all* chiral primaries of the CFT using the solutions for the R sector found in [5]. From this set we see that the 'conical defect metrics' (case (a)) are a special case[2] where all the n_i in (1.2) are *equal*. We then find that the Aichelburg-Sexl type metrics (case (b)) arise from chiral primaries where a fraction of the n_i in (1.2) are unity (σ_1^{--} is just the identity operator) while the rest are large numbers but not necessarily equal.

Both the above metrics describe chiral primaries of the D1-D5 bound state; the two cases give in some sense two opposite limits of the general metrics dual to chiral primaries. By contrast, the giant graviton metrics turn out to describe D1-D5 systems that are 'disassociated' – some fraction of the D1 and D5 branes are pulled out from the bound state to a finite distance away from the other branes.

In more detail, our methods and results are as follows:

(a) We first recall the construction of [5] that gives the metrics for general R sector ground states. We also recall the notion of [7, 8] that spectral flow to the NS sector is accomplished by a 'large change' of coordinates which returns us to the same metric but with a different periodicity for the fermions. Thus we obtain the metrics for general chiral primary states; for special families of states this construction was considered in [9].

(b) We then consider two special families of metrics: the conical defect metrics considered in [7, 8] and the Aichelburg-Sexl type metric constructed in [9]. The chiral primary state corresponding to the conical defect metrics was constructed in [5], and was given by the action of twist operators on the NS vacuum with all twist operators having the same order and spin. The CFT description for the Aichelburg-Sexl metric was not known, and we find this description here; we map the D1-D5 state by S,T dualities to a fundamental string carrying vibrations and then, following the general analysis of [5], read off the orders and spins of the twist operators in the CFT state from the Fourier transform of the vibration profile.

(c) We note that we can separate a D1 brane from the D1-D5 bound state at special values of the moduli, and displace it away from the rest of the bound state in a transverse direction. This is a static configuration, but after the coordinate change to the NS sector the separated D1 brane is seen to spin around the S^3 giving a giant graviton. We get a unified picture for giant gravitons on the AdS_3 and on the S^3, as both arise as special limits of a giant graviton that winds on both the AdS_3 and the S^3.

Using this picture for giant gravitons we can construct the metric for any distribution of giant gravitons, since the classical metric produced by a set of disjoint D1-D5 systems can be written down by just superposing the harmonic functions appearing in the metric. As an example we take a particular distribution of giant gravitons and compute the metric it produces.

Given this construction of giant gravitons, we suggest that giant graviton excitations do not describe chiral primary states of the D1-D5 system, since such chiral primaries should be given by configurations of the D1-D5 *bound* state; the metrics for all such bound states can be obtained as in (a) above.

[2] This subclass of metrics was discussed (from the viewpoint of the Ramond (R) sector) in [7, 8].

(d) We return to the metrics for generic chiral primary states and analyze the large r behavior of the first few corrections to the background $AdS_3 \times S^3$. We note that the conical defect metrics are locally exactly $AdS_3 \times S^3$, so all excitations save some global pure gauge fields are zero at infinity. For the Aichelburg-Sexl type metric we find nontrivial corrections at infinity, including the excitation of some low mass scalars and gauge fields which we study explicitly. We conjecture that the existence of these fields is related to the fact that the twist operators creating the dual CFT state have a wide dispersion in their orders in the Aichelburg-Sexl case, while they have no dispersion in the conical defect metrics. Following up on this conjecture we look at the asymptotic fields for generic chiral primary states, and find that we can relate the existence of these corrections to the presence of dispersions in the orders and spins of the twist operators creating the dual CFT state.

2 The metrics for general chiral primaries

2.1 The D1-D5 and FP systems

Let us recall some results from [10, 5, 9]. We take $M_4 = T^4$ for concreteness. Consider type IIB string theory compactified on $T^4 \times S^1$. The T_4 has volume $(2\pi)^4 V$ and is parametrized by coordinates z_1, z_2, z_3, z_4. The S^1 has radius $2\pi R$ and is parametrized by a coordinate y. The noncompact directions are x_0, x_1, x_2, x_3, x_4. We wrap n_5 D5 branes on $T^4 \times S^1$ and n_1 D1 branes on S^1. We are interested in the CFT arising from the low energy physics of the *bound state* of these branes.

The field theory describing this bound state has its fermions periodic around the S^1 and is thus in the Ramond (R) sector.[3] The R ground state of the CFT is highly degenerate, with a degeneracy $\sim e^{2\sqrt{2}\pi\sqrt{n_1 n_5}}$.

One might naively think that such bound states are pointlike in the transverse space x_1, x_2, x_3, x_4, and that the corresponding metrics have a pointlike singularity at the origin $r = 0$ in this space. To see that such is not the case and to construct the actual metrics for the D1-D5 ground states, it is helpful to map the D1-D5 system by a set of S, T dualities to the FP system: we obtain a fundamental string (F) wrapped n_5 times around S^1, carrying momentum (P) along the S^1. The fact that we are studying a *bound* state of these charges implies that the F string is a single 'multiwound' string, and all the momentum P is carried by traveling waves on this string.

It is possible to write down the supergravity solution for a string carrying traveling waves in one direction. The only subtlety is that we have many strands of the 'multi-wound' string, and must superpose harmonic functions arising from different strands.[4]

[3]The periodicity of the worldvolume fermions in induced from the periodicity of fermions in the bulk, and in the bulk the fermions must be periodic in order that the cosmological constant vanish and flat spacetime be a solution at infinity.

[4]The different stands do not all carry the same profile of vibration, since the only restriction on the

Such a superposition was carried out in [10] for the case where the string profile was a uniform helix; dualizing back to the D1-D5 system we obtained the subclass of metrics studied in [7, 8, 11]. More generally, let the F string have a vibration profile in the noncompact spatial directions[5] described by $x_i = F_i(v)$, where $i = 1, 2, 3, 4$ and $v = t - y$. The supergravity solution created by such a string is given in eqn. (A.1) in Appendix A. If we perform the chain of dualities that take us to the D1-D5 system then we get the supergravity solution

$$\begin{aligned} ds^2 &= \sqrt{\frac{H}{1+K}}\Big[-(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2\Big] + \sqrt{\frac{1+K}{H}} d\vec{x}\cdot d\vec{x} \\ &+ \sqrt{H(1+K)} d\vec{z}\cdot d\vec{z} \qquad (2.1) \\ e^{2\Phi} &= H(1+K), \qquad C^{(2)}_{ti} = \frac{B_i}{1+K}, \qquad C^{(2)}_{ty} = -\frac{K}{1+K}, \\ C^{(2)}_{iy} &= -\frac{A_i}{1+K}, \qquad C^{(2)}_{ij} = C_{ij} + \frac{A_i B_j - A_j B_i}{1+K} \qquad (2.2) \end{aligned}$$

The functions H, K and A_i appearing in this solution are related to the profile $\mathbf{F}(v)$:

$$H^{-1} = 1 + \frac{Q}{l}\int_0^l \frac{dv}{(\vec{x} - \vec{F})^2}, \quad K = \frac{Q}{l}\int_0^l \frac{|\dot{F}|^2 dv}{(\vec{x} - \vec{F})^2}, \quad A_i = -\frac{Q}{l}\int_0^l \frac{\dot{F}_i dv}{(\vec{x} - \vec{F})^2} \qquad (2.3)$$

and the forms B_i and C_{ij} are defined by the duality relations:

$$dB = -{}^*dA, \qquad dC = -{}^*dH^{-1}. \qquad (2.4)$$

(The form $C^{(2)}$ is the Ramond-Ramond B field B^{RR}_{ij}.)

Different profiles of the vibrating F string correspond to different R ground states of the D1-D5 system. (This relation was studied in detail in [5] and will be reviewed below.) It is important that the F string has no longitudinal vibration mode, and must therefore necessarily bend away from the central axis in order to carry the momentum P. The corresponding metrics thus have an extended singularity rather than a pointlike singularity at $r = 0$.

2.2 Spectral flow

As mentioned above, the above metrics describe states of the R sector, since the fermions are periodic under $y \to y + 2\pi R$. We wish to obtain metrics for the NS sector, where

vibration is that it close after n_5 turns around the S^1. But as long as the momentum flows along the same direction on each strand we can superpose harmonic functions, and the final configuration is a 1/4 BPS solution of IIB supergravity.

[5]We do not consider vibrations in the T^4 directions, though in principle these could be handled the same way. The restriction on vibrations corresponds, after mapping to the D1-D5 system, to using cohomology elements $h_{00}, h_{02}, h_{20}, h_{22}$ from T^4. Both T^4 and $K3$ have the one form of each of these types; the remaining vibrations would differentiate between these two spaces.

we have chiral primaries ($h=j$, $\bar{h}=\bar{j}$). The operation of spectral flow maps R sector ground states in the CFT to chiral primaries in the NS sector of the CFT, in a 1-1 and onto fashion. For the special class of D1-D5 metrics studied in [7, 8] the description of spectral flow in the supergravity dual was discussed, and we can extend that to the general set of metrics (2.1) as follows.

We first take the limit

$$R \gg (Q_1Q_5)^{1/4} \gg a, \; |x| \ll (Q_1Q_5)^{1/4}, \tag{2.5}$$

where a is the size of the singularity ($|\mathbf{F}| < a$). Then we should put the following expressions for H^{-1} and $1+K$ (since unity can be ignored in comparison to the term of the form $\sim 1/x^2$)

$$H^{-1} = \frac{Q}{l}\int_0^l \frac{dv}{(\vec{x}-\vec{F})^2}, \quad 1+K = \frac{Q}{l}\int_0^l \frac{|\dot{F}|^2 dv}{(\vec{x}-\vec{F})^2} \tag{2.6}$$

in (2.1) and we get another set of exact solutions to supergravity. The region

$$a \ll x \ll (Q_1Q_5)^{1/4} \tag{2.7}$$

is the large radius region of anti-de-Sitter space and here the metric is locally $AdS_3 \times S^3 \times T^4$. To see this explicitly, consider the four coordinates x_1, x_2, x_3, x_4 and map these to a set of polar coordinates

$$\begin{aligned} x_1 &= \tilde{r}\sin\tilde{\theta}\cos\tilde{\phi}, \quad x_2 = \tilde{r}\sin\tilde{\theta}\sin\tilde{\phi}, \\ x_3 &= \tilde{r}\cos\tilde{\theta}\cos\tilde{\psi}, \quad x_4 = \tilde{r}\cos\tilde{\theta}\sin\tilde{\psi} \end{aligned} \tag{2.8}$$

In the region (2.7) the 3-dimensional surface of constant $\tilde{r}$ is an S^3 with proper radius $(Q_1Q_5)^{1/4}$

$$ds^2 = (Q_1Q_5)^{1/4}[d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\phi}^2 + \cos^2\tilde{\theta}d\tilde{\psi}^2] \tag{2.9}$$

The fermions have charge $(\frac{1}{2}, \frac{1}{2})$ under the $SU(2)\times SU(2) \approx SO(4)$ symmetry group of the S^3. In the 1+1 dimensional CFT we can go from the R to the NS sector if we add an extra flat connection A_y with $e^{i\int A\cdot dl} = -1$ in each $SU(2)$. Such a connection can be induced by a coordinate transformation which rotates the S^3 as we move along the y circle. In the supergravity dual such a rotation of the S^3 induces the map

$$\psi = \tilde{\psi} - \frac{y}{R} \qquad \phi = \tilde{\phi} - \frac{t}{R} \tag{2.10}$$

This generates the appropriate connection A, and the fermions may now be considered to be in the NS sector.[6] For the subclass of metrics in [7, 8] the corresponding NS sector metrics were studied in [9]. But we may make the same coordinate change at infinity also for the more general class (2.1), and thus obtain metrics dual to all chiral primary states of the NS sector.

[6]In the NS sector we will often write $\chi = y/R$.

2.3 The detailed FP $\to$ D1-D5 map

We recall in more detail the map [5] between 1/4 BPS states of the FP system and chiral primaries of the D1-D5 system. The traveling waves travel in only one direction along the F string, for states with supersymmetry. The string has a total length $2\pi R' n_5$, where R' is the radius of the S^1. On this string the traveling wave can be described by specifying the number of quanta in each harmonic. Creating a quantum in the nth harmonic implies in the D1-D5 system the action of the twist operator σ_n on $|0\rangle_{NS}$:

$$a_n^{\dagger,i} \;\to\; \sigma_n^{\epsilon,\epsilon'} \tag{2.11}$$

Here $i = 1, 2, 3, 4$ labels the four possible polarizations of the vibration mode in the four noncompact directions x_1, x_2, x_3, x_4. We can write $SO(4) \approx SU(2) \times SU(2)$ and thus re–express this vector index i in terms of a pair of spinor indices $\epsilon = \pm, \epsilon' = \pm$ which describe the spin $(\frac{1}{2}, \frac{1}{2})$ representation of $SU(2) \times SU(2)$. In the D1-D5 system in the NS sector we have four chiral primaries for each twist σ_n :

$$\sigma_n^{--}, \quad h = j_3 = \frac{n-1}{2}, \ \bar{h} = \bar{j}_3 = \frac{n-1}{2} \tag{2.12}$$

$$\sigma_n^{+-}, \quad h = j_3 = \frac{n+1}{2}, \ \bar{h} = \bar{j}_3 = \frac{n-1}{2} \tag{2.13}$$

$$\sigma_n^{-+}, \quad h = j_3 = \frac{n-1}{2}, \ \bar{h} = \bar{j}_3 = \frac{n+1}{2} \tag{2.14}$$

$$\sigma_n^{++}, \quad h = j_3 = \frac{n+1}{2}, \ \bar{h} = \bar{j}_3 = \frac{n+1}{2} \tag{2.15}$$

These twist operators are thus also labelled by a set $\epsilon = \pm, \epsilon' = \pm$, The map (2.11) says that we should identify the indices $(\pm, \pm)$ on the σ_n with the indices $(\pm, \pm)$ obtained from the vector index i on the $a_n^{\dagger,i}$. The reason for this is the following. The S,T dualities relate the F string carrying the vibrations $a_n^{\dagger,i}$ to the D1-D5 system in the R sector. If we spectral flow the chiral primaries $\sigma_n^{\epsilon,\epsilon'}$ from the NS to the R sector then we find that they have $SU(2) \times SU(2)$ spins

$$j_3 = \frac{\epsilon}{2}, \quad \bar{j}_3 = \frac{\epsilon'}{2} \tag{2.16}$$

and they form the $(\frac{1}{2}, \frac{1}{2})$ representation of $SU(2) \times SU(2)$. The spins of the vibrations on the F string and the spins in the R sector of the D1-D5 system are immediately identified with each other, since the S,T dualities do not affect the transverse coordinates x_1, x_2, x_3, x_4 which give the $SO(4) \approx SU(2) \times SU(2)$.

We can get a classical vibration profile for the string if we excite a large number of vibration quanta. Such a state maps to a state of the form (1.2) in the NS sector of the D1-D5 system.

3 Metrics for special cases of interest

The solutions (2.1) possess a singularity at the location of the F string in the FP system, which maps to a corresponding singularity in the D1-D5 system. It is important to note that this singularity is an acceptable one that will be resolved in the full string theory. In the FP system we have started with a fundamental string in flat space, and this string is a valid quantum in the full string theory. Thus the metric it creates is a true solution of the full theory. The dualities we have used to go to the D1-D5 system are also true dualities since they are made along closed cycles of the T^4; thus they are not just 'solution generating techniques' for classical supergravity solutions, which would be the case for example if we dualized along some other direction of the spacetime. In performing this duality we have 'smoothed over' the singular locus of the F string since this string had a large number of strands (this is explained in detail in [5]); this smoothing should generate the correct classical D1-D5 solution for the R sector. The truncation to the AdS region was done by a limiting process $R \gg (Q_1Q_5)^{1/4}$ and looking only at $r \ll (Q_1Q_5)^{1/4}$; this is therefore also an allowed step in the full quantum string theory. Lastly, map to the NS sector was just a change of coordinates (2.10), and so does not spoil the existence of the solution. Thus the metrics we make for the chiral primaries are the classical limits of exact string theory solutions, and not just classical solutions with singularity.

3.1 Case I: $AdS_3 \times S^3$

Let the F string carry a traveling wave which has all its energy only in the lowest harmonic. Further, let the polarization of this wave be such that the string rotates in a helical fashion in the $x_1 - x_2$ plane (recall that x_1, x_2, x_3, x_4 are the four transverse direction in which we are considering vibrations). It was shown in [10] that this configuration of the FP system possesses the maximal possible angular momentum for given F and P charges: under the $SU(2) \times SU(2)$ this configuration has $(j, j') = (\frac{n_1 n_5}{2}, \frac{n_1 n_5}{2})$. The metric of this configuration was computed, and was shown to map under dualities to the 'maximally rotating' D1-D5 bound state discussed in [7, 8].

In more detail, consider a FP metric generated by a following profile:

$$G_1 = a\cos\omega v', \quad G_2 = a\sin\omega v', \quad G_3 = G_4 = 0 \tag{3.1}$$

We choose

$$\omega = \frac{1}{n_5 R'} \tag{3.2}$$

which corresponds to the F string having all its energy in the lowest harmonic (recall that the F string has winding number n_5 around the y circle, and we use R' to denote the radius of y direction on the FP side).

We wish to write the metric for the corresponding D1-D5 system. In terms of the parameters of the D1–D5 system the profile (3.1) becomes:

$$F_1 = a\cos\omega v, \quad F_2 = a\sin\omega v, \quad F_3 = F_4 = 0, \quad \omega = \frac{R}{n_5} \tag{3.3}$$

For later use it is convenient to introduce

$$l = 2\pi R' n_5 = \frac{2\pi n_5}{R} \tag{3.4}$$

The D1-D5 metric is given by eqn. (2.1), with coefficient functions H^{-1}, K, A_i given by (2.3). As an example of the computation note that in H^{-1} we get the harmonic function created by a uniform circular source:

$$\begin{aligned} H^{-1} &= 1 + \frac{Q}{2\pi}\int_0^{2\pi} \frac{d\xi}{(x_1 - a\cos\xi)^2 + (x_2 - a\sin\xi)^2 + x_3^2 + x_4^2} \\ &= 1 + \frac{Q}{\sqrt{(\tilde{r}^2 + a^2)^2 - 4a^2\tilde{r}^2\sin^2\tilde{\theta}}} \end{aligned} \tag{3.5}$$

where in the last step we have used the polar coordinates (2.8). The above expression simplifies if we change from $\tilde{r}, \tilde{\theta}$ to coordinates r, θ:

$$\tilde{r} = \sqrt{r^2 + a^2\sin^2\theta}, \quad \cos\tilde{\theta} = \frac{r\cos\theta}{\sqrt{r^2 + a^2\sin^2\theta}} \tag{3.6}$$

($\tilde{\phi}$ and $\tilde{\psi}$ remain unchanged). Then we get

$$H^{-1} = 1 + \frac{Q}{r^2 + a^2\cos^2\theta} \tag{3.7}$$

We can go to the near horizon limit ($r \ll (Q_1Q_5)^{1/4}$), and then the metric of the D1–D5 system becomes[7] (we write $r' = r/a$):

$$\begin{aligned} ds^2 &= -(r'^2 + 1)\frac{a^2dt^2}{\sqrt{Q_1Q_5}} + r'^2\frac{a^2dy^2}{\sqrt{Q_1Q_5}} + \sqrt{Q_1Q_5}\frac{dr'^2}{r'^2 + 1} \\ &+ \sqrt{Q_1Q_5}\left[d\theta^2 + \cos^2\theta\left(d\tilde{\psi} - \frac{ady}{\sqrt{Q_1Q_5}}\right)^2 + \sin^2\theta\left(d\tilde{\phi} - \frac{adt}{\sqrt{Q_1Q_5}}\right)^2\right] \end{aligned} \tag{3.8}$$

The charge Q_1 is related by dualities to the momentum charge P carried by the F string, and for the profile (3.3) we find

$$Q_1 = a^2\omega^2 Q_5, \tag{3.9}$$

which for the profile we are considering translates into

$$\frac{a}{\sqrt{Q_1Q_5}} = \frac{1}{R}. \tag{3.10}$$

Lastly, performing the coordinate change (2.10) that gives spectral flow we get the metric

$$\begin{aligned} ds^2 &= \sqrt{Q_1Q_5}\left[-(r'^2 + 1)\frac{dt^2}{R^2} + r'^2\frac{dy^2}{R^2} + \frac{dr'^2}{r'^2 + 1}\right] \\ &+ \sqrt{Q_1Q_5}\left[d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2\right], \end{aligned} \tag{3.11}$$

[7] The complete asymptotically flat metric is given for example in [10].

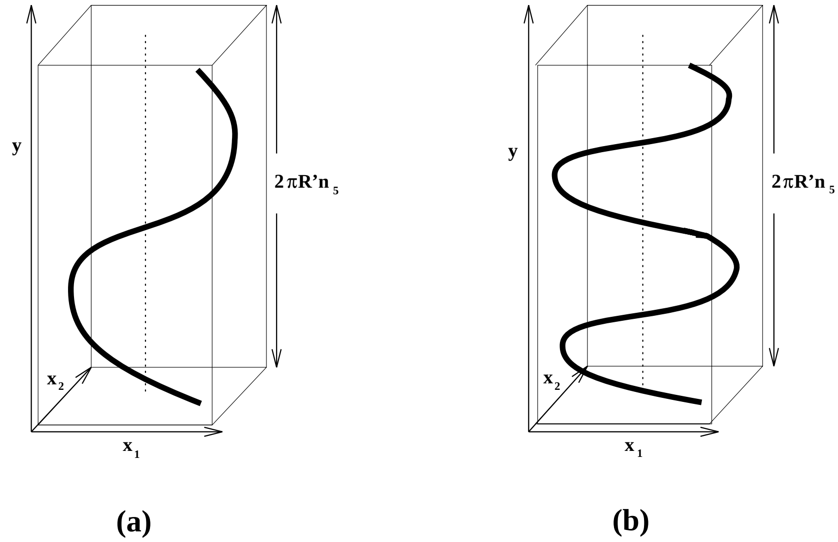

Figure 1: (a) The F string opened up to its full length, rotating in a helix with a single turn; this gives $AdS_3 \times S^3$. (b) The F string executing a helix in the second harmonic; this gives a conical defect metric.

which is just $AdS_3 \times S^3 \times T^4$.

On the CFT side, the fact that all the energy of the FP system is in the lowest harmonic implies under the map (2.11) that all the twist operators have order unity. The choice of rotation plane $(x_1 - x_2)$ implies that we get the operator σ_1^{--} for each quantum of vibration. But this is just the identity operator, with $h = j = \bar{h} = \bar{j} = 0$. Thus the CFT state dual to the metric under consideration is (in the NS sector) just $|0\rangle_{NS}$.

In Figure 1(a) we have sketched the F string, opened up to its full length $2\pi R' n_5$ and carrying the vibration profile mentioned above. The dual state $|0\rangle_{NS}$ is of course the simplest chiral primary. We will use similar figures to discuss more complicated cases below.

3.2 Case II: The conical defect metrics

Take the F string to again rotate in the $x_1 - x_2$ plane but let all the energy be in the mth harmonic (instead of the first harmonic). Figure 1(b) shows the F string carrying such a vibration with $m = 2$. The metric for this configuration yields [10], after dualization to the D1-D5 system, the general metrics discussed in [7, 8]. After the limit (2.5) is taken, we get a metric that is locally $AdS_3 \times S^3 \times T^4$. But there is a singular circle, such that at any point along this circle we have a 'conical defect': a wedge is removed from the AdS_3 and the S^3 is glued across the cut after a suitable rotation. Taking the limit (2.5), and making the coordinate changes (2.8), (3.6), (2.10) just as in Case I above we get the metrics for the chiral primary [9]:

$$\begin{aligned} ds^2 &= \sqrt{Q_1Q_5}\left[-(r'^2+\gamma^2)\frac{dt^2}{R^2}+r'^2\frac{dy^2}{R^2}+\frac{dr'^2}{r'^2+\gamma^2}\right] \\ &+ \sqrt{Q_1Q_5}\left[d\theta^2+\cos^2\theta\left(d\psi+(1-\gamma)\frac{dy}{R}\right)^2+\sin^2\theta\left(d\phi+(1-\gamma)\frac{dt}{R}\right)^2\right], \end{aligned} \tag{3.12}$$

where $\gamma = 1/m$.

In the CFT we find that the dual state is (using the map (2.11))

$$[\sigma_m^{--}]^{N/m}|0\rangle_{NS}, \tag{3.13}$$

where $N = n_1 n_5$. The number N/m of twist operators comes from the fact that all the N copies of T^4 in the symmetric product must be involved in the permutation (a copy not involved in the permutation gives a permutation cycle of length unity, which gives the twist σ_1 and implies a vibration quantum in the *first* harmonic on the F string).

3.3 Case III - the Aichelburg-Sexl metric

Consider the D1-D5 geometry $AdS_3 \times S^3 \times T^4$; this is the dual of the NS vacuum $|0\rangle_{NS}$. As discussed in section 1, we can get a chiral primary with $h = j = \bar{h} = \bar{j}$ by taking massless quanta at the center of AdS_3, rotating on the S^3. The angular momentum on the S^3 is described by the spin state under the group $SO(4) \approx SU(2) \times SU(2)$, and we should take $|j, j_3\rangle = |j, j\rangle$, $|\bar{j}, \bar{j}_3\rangle = |j, j\rangle$ to get the chiral primary. This implies that the quantum rotates along the direction ϕ at $\theta = \pi/2$.

We are interested in the metric obtained after we take the backreaction of the rotating quanta into account. To reach the classical limit we take several such massless quanta rotating along the ϕ circle at $\theta = \pi/2$. These quanta can have different values for j, but the overall distribution is selected to satisfy the following conditions:

(i) We take a large number of quanta so that we can approximate the energy distribution by a continuous one along the circle $\theta = \pi/2$.

(ii) We let the distribution of quanta be uniform along the ϕ coordinate, so that the density of mass and energy is uniform along the circle $\theta = \pi/2$.

(iii) We let each $j \gg 1$, so that the quanta have wavelength $\lambda \ll R_{AdS}$, where $R_{AdS} = (Q_1Q_5)^{1/4}$ is the radius of the AdS_3 and the S^3. This allows us to treat the quanta as pointlike.

Near this line of massless quanta $\theta = \pi/2, r = 0$ we should find the metric produced by a uniform line of massless particles in 5+1 flat spacetime (we always smear all wavefunctions uniformly over the T^4). The latter distribution generates the metric

$$ds^2 = -dt^2 + dz^2 + \frac{q}{(x_i x_i)}(dt-dz)^2 + \sum_{i=1}^{4} dx_i dx_i \tag{3.14}$$

We thus need a solution to the full supergravity equations that behaves as (3.14) near the singular line and goes over to $AdS_3 \times S^3 \times T^4$ at large r. There is no obvious way to find such a solution. The general solutions (2.1) constructed above started with a given profile for the F string, while in the present case we want a D1-D5 metric but do not know which traveling wave on the F string will generate the answer. But the Aichelburg-Sexl solution in asymptotically flat space has an interesting property: the linearized gravity solution turns out to also be an exact solution. In the present case one can find the linearized solution with some effort, and again this linearized solution turns out to be exact. This procedure was carried out in [9] and the following solution was obtained for the line of massless particles[8]:

$$\begin{aligned} ds^2 &= L^2\left\{-(1+r'^2)d\tau^2 + \frac{dr'^2}{1+r'^2} + r'^2 d\chi^2 + d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2\right\} + \sum_{i=1}^{4} dz_i dz_i \\ &+ \frac{qL^2}{r'^2+\cos^2\theta}\left[\left\{(1+r'^2)d\tau - \sin^2\theta d\phi\right\}^2 - \left\{r'^2 d\chi - \cos^2\theta d\psi\right\}^2\right], \end{aligned} \tag{3.15}$$

the R–R two–form field and dilaton are:

$$\begin{aligned} B^{RR} &= L^2 e^{-\Phi_0}\cos^2\theta d\phi \wedge d\psi + r'^2 L^2 e^{-\Phi_0} d\tau\wedge d\chi \\ &- \frac{qL^2 e^{-\Phi_0}}{r'^2+\cos^2\theta}\left\{(1+r'^2)dt - \sin^2\theta d\phi\right\}\wedge\left\{r'^2 d\chi - \cos^2\theta d\psi\right\} \end{aligned} \tag{3.16}$$

$$e^{2\Phi} = e^{2\Phi_0}. \tag{3.17}$$

Here $e^{2\Phi_0}$ is an arbitrary constant which we will later interpret as the ratio of the brane charges: $e^{2\Phi_0} = Q_1/Q_5$.

Looking at this D1-D5 geometry we cannot directly see what state in the CFT it is dual to. What we should do now is to find an FP solution that is the dual of this D1-D5 solution; the Fourier analysis of the FP solution using the relation (2.11) will then tell us which chiral primary the solution corresponds to.

To find the profile $\mathbf{F}(v)$ one should perform several steps. First one should go from the solution in the NS sector (3.15)–(3.17) to the solution in the Ramond sector by making a change of coordinates inverse to (2.10). From the resulting solution (which is asymptotically $AdS_3 \times S^3$) one can read off the values of the harmonic functions, and the extension to the asymptotically flat space is achieved by adding a constant to H^{-1}. After performing these steps we get:

$$H^{-1} = 1 + \frac{L^2 e^{-\Phi_0}}{r^2+a^2\cos^2\theta}, \quad K = \frac{L^2 e^{\Phi_0}}{r^2+a^2\cos^2\theta}, \quad A_\phi = \sqrt{1-q}\frac{L^2 a\sin^2\theta}{r^2+a^2\cos^2\theta} \tag{3.18}$$

Here we introduced a new coordinate

$$r = \sqrt{1-q}\; Lr' \tag{3.19}$$

[8]We recall, that since we are in the NS sector, the coordinates on the sphere are called θ, ϕ, ψ. In (3.15) we also use coordinates τ and χ which are related with t and y: $t = \tau R$, $y = \chi R$.

which becomes a radial coordinate at the flat infinity, and the parameter

$$a = \sqrt{1-q}\ L \tag{3.20}$$

Let us now discuss how we can get the harmonic functions (3.18) from the profile of a vibrating string.

As a warm up example, first consider the vibration profile of the F string shown in Fig. 2(a). We have seen that the total length of this string is $2\pi R' n_5$. A part of this string (characterized by a fraction $\xi < 1$) executes a single turn around a uniform helix in the $x_1 - x_2$ plane. The remainder of the string is at a constant location in the x_1, x_2, x_3, x_4 space. Thus the profile $F_i(v)$ is

$$\mathbf{F}(v) = a\mathbf{e}_1 \cos\left(\frac{2\pi m v}{\xi l}\right) + a\mathbf{e}_2 \sin\left(\frac{2\pi m v}{\xi l}\right), \quad 0 \le v < l\xi \tag{3.21}$$

$$\mathbf{F}(v) = a\mathbf{e}_1, \quad l\xi \le v < l, \qquad l = \frac{2\pi n_5}{R} \tag{3.22}$$

This profile gives rise to the coefficient functions

$$H^{-1} = 1 + \frac{Q\xi}{r^2 + a^2\cos^2\theta} + \frac{Q(1-\xi)}{(x_1 - a)^2 + x_2^2 + x_3^2 + x_4^2}, \tag{3.23}$$

$$A_\phi = \frac{a^2 Q}{r^2 + a^2\cos^2\theta}\frac{2\pi}{l}\sin^2\theta, \quad K = \frac{Qa^2}{r^2 + a^2\cos^2\theta}\frac{1}{\xi}\left(\frac{2\pi}{l}\right)^2 \tag{3.24}$$

Now consider the profile of the F string depicted in Fig 2(b). The helical part and the straight part in Fig 1(a) have each been broken up into several segments and such segments alternate: thus the string describes a part of a helix, then stays at a constant x_i location for some length, then again describes a part of the helix, and so on. The locations and lengths of the straight line segments are arbitrary, except that we impose the following overall conditions:

(i') We take a large number of straight line segments so that we can approximate their distribution by a continuous one along the circle $x_1^2 + x_2^2 = constant$.

(ii') We let the distribution of straight line segments be uniform (in a coarse grained sense) along this circle.

(iii') We let each straight line segment have a length $l \gg 2\pi R'$ so that it describes several windings around the compactified circle y.

For such an F string we get a change in the last term in H^{-1} in (3.23). The location $(a, 0, 0, 0)$ in (3.22) for the straight part of the F string is now smeared uniformly over a circle of radius a in the $x_1 - x_2$ plane. In the harmonic function H^{-1} we get a linear superposition of the effects of different parts of the F string, and we find that with this

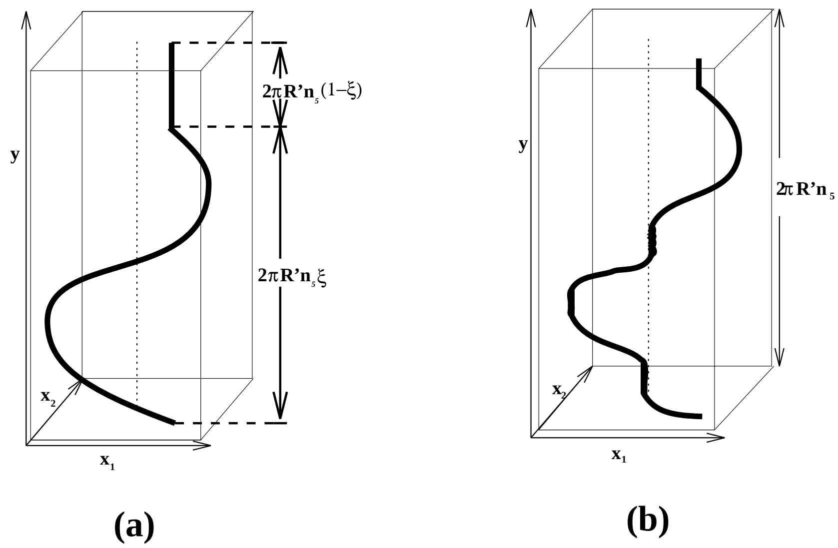

Figure 2: (a) The F string executes one turn of a helix and then stays at a constant location. (b) The constant part of the string is broken into several segments which are interspersed with the curved part; this gives the Aichelburg-Sexl solution.

smearing we need to take the average of the last term in H^{-1} over the locations of the straight line segments. We get

$$H^{-1} = 1 + \frac{Q}{r^2 + a^2\cos^2\theta} \tag{3.25}$$

and similarly all the other coefficient functions (3.18) with following identification of the parameters:

$$L = (Q_1 Q_5)^{1/4}, \quad e^{2\Phi_0} = \frac{Q_1}{Q_5}, \quad Q_5 = Q, \quad Q_1 = \frac{Qa^2}{\xi}\left(\frac{2\pi}{l}\right)^2 \tag{3.26}$$

Having obtained the profile of the F string which gives the Aichelburg-Sexl solution for the D1-D5 system we can find the corresponding chiral primary in the CFT by the map (2.11). The conditions (i'), (ii'), (iii') correspond respectively to the requirements (i), (ii), (iii) above. In particular (iii') implies that the straight line segments give rise to chiral primaries which have twist operators σ_n with $n \gg 1$. The fact that the curved part of the string describes only one cycle of the helix implies that it contributes only copies of σ_1^{--} which is just the identity operator and thus does not affect the NS vacuum. Thus the state of the dual CFT has the form

$$[\sigma_1^{--}]^p \sigma_{n_1}^{--}\sigma_{n_2}^{--}\ldots\sigma_{n_k}^{--}|0\rangle_{NS} \;=\; \sigma_{n_1}^{--}\sigma_{n_2}^{--}\ldots\sigma_{n_k}^{--}|0\rangle_{NS} \tag{3.27}$$

with all $n_i \gg 1$. Note that $\sum_i n_i + p = N$.

We thus see that the Aichelburg-Sexl metric, which has no conical defect, also describes chiral primaries, but these chiral primaries are rather different from those of the type (3.13). In (3.13) all order n_i of the twists are equal, so the dispersion $\langle n_i^2\rangle - \langle n_i\rangle^2 = 0$. By contrast, in (3.27) we have $\langle n_i^2\rangle - \langle n_i\rangle^2$ of order $\langle n_i\rangle^2$, where we have taken into account the fact that a significant number p of the indices involved in the permutations are in permutation cycles of length $n_i = 1$ – i.e., they are left untouched by the permutation.

4 Giant gravitons

We have seen above that both the conical defect metrics of [7, 8] and the Aichelburg-Sexl type metric found in [9] are special cases of the metrics that were obtained for generic chiral primaries in [5]. We now proceed to examine giant gravitons and the metrics they would create by their backreaction on the geometry.

Giant gravitons exist in $AdS_3 \times S^3 \times T^4$ only for a subset of the possible values of the moduli; we can get giant gravitons for example if we set all gauge potentials like the NS B field to be zero on T^4. Let us assume that we have chosen such a spacetime. As before we smear all fields on the T^4, and consider the 5+1 spacetime $AdS_3 \times S^3$. We can get giant gravitons where the D1 brane expands in this D1-D5 geometry, and also when the D5 brane expands – the D5 brane wrapped on T^4 is also a string in the 5+1 spacetime. (We can also get all possible bound states of these two kinds of giant gravitons, as we will see below.)

For $AdS_3 \times S^3$ the metric and RR B field are

$$ds^2 = -\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\chi^2 + d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2 \tag{4.1}$$

$$B^{RR} = \cos^2\theta d\phi \wedge d\psi + \sinh^2\rho d\tau \wedge d\chi \tag{4.2}$$

Consider a 'giant graviton' made of the threshold bound state of m_1 D1 branes and m_5 D5 branes. The DBI + Chern Simons action is (recall that the H^{RR} field strength is self-dual in 5+1 dimensions)

$$\begin{aligned} S &= -(m_1T_1 + m_5T_5V)\int d^2\sigma\sqrt{-G_{MN}\partial X^M\partial X^N} \\ &+ \frac{m_1T_1 + m_5T_5V}{2}\int d^2\sigma\epsilon^{ab}B^{RR}_{MN}\partial_aX^M\partial_bX^N \end{aligned} \tag{4.3}$$

We let the giant graviton run around the sphere along the coordinate ϕ. In other spacetimes of the form $AdS_m \times S^n$ we have two kinds of giant gravitons: those which expand on the S^n, and those which expand on the AdS_m. In the present case we see however that we can have a more general giant graviton, which has a nonzero size in *both* $AdS_3 \times S^3$. Letting σ^0, σ^1 be the world sheet coordinates of the giant graviton we write the ansatz

$$\tau = \sigma^0, \quad \chi = \sigma^1, \quad \psi = \pm\sigma^1, \quad \rho = \bar\rho, \qquad \theta = \bar\theta, \qquad \phi = \phi(\tau). \tag{4.4}$$

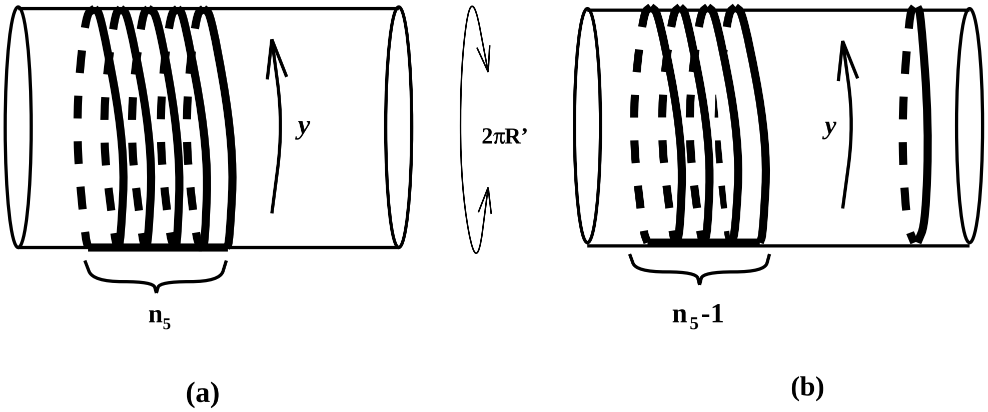

Figure 3: (a) In all bound states the F string closes after n_5 turns around the y direction. (b) One turn of the F string is broken off; this gives a giant graviton.

This giant graviton winds once around the χ circle in AdS_3 and also once around the ψ circle in S^3, rotates along the ϕ direction, and maintains a constant radius $\bar{\rho}$ on AdS_3 and a constant radius $\bar{\theta}$ on S^3. Extremsing the action yields a solution with

$$\phi = \pm\tau = \pm t \tag{4.5}$$

so that the giant graviton rotates with unit angular velocity. The choice of sign $\pm$ must be the same in (4.4), (4.5). This choice of signs stems from the fact that in $AdS_3 \times S^3$ we can reverse the sign of both angular coordinates ϕ, ψ while leaving the B^{RR} field unchanged, so this sign change is a symmetry of the giant graviton DBI action on $AdS_3 \times S^3$. The radii $\bar{\rho}, \bar{\theta}$ are *arbitrary*, so that the size of the giant graviton in AdS_3 and in S^3 does not depend on the angular momentum it carries; in fact the angular momentum is determined from the charges m_1, m_5. (The fact that the effective potential governing the radial size of the giant graviton is flat in $AdS_3 \times S^3$ was noted in the early papers on giant gravitons [6].)

Note that if we take the limit $\bar{\rho} \to 0$ then we get a giant graviton expanding on S^3 only, and if we take $\bar{\theta} \to \pi/2$ then we get a giant graviton expanding on AdS_3 only.

4.1 The FP representation of a giant graviton

In subsection 3.1 we had seen that the spacetime $AdS_3 \times S^3$ was obtained under S,T dualities from a particular configuration of the FP system – the F string carried a traveling wave in a form of a uniform helix executing a single turn. Let us now see if we can find the FP representation of the spacetime $AdS_3 \times S^3$ which has present in it a single giant graviton which we regard as a 'test particle'; i.e., for the moment we ignore its backreaction on the geometry.

Figure 3(a) depicts the F string winding n_5 times around the y circle before closing on itself; this is the behavior of the F string for all bound states of the system. In Figure 3(b) we have depicted *two* separate F strings.[9] The first F string winds $n_5 - 1$ times around the y direction, and if we open it up to its full length, it will have a vibration profile similar to that of Fig 1(a) – thus by itself this string would generate the spacetime $AdS_3 \times S^3 \times T^4$ after dualities. The second F string winds just once around the y circle and carries no vibrations; it is thus much lighter than the first string and for the moment we regard it as a test string in the background produced by the first F string. The center of the helix produced by the first string is at $x_i = 0, i = 1, 2, 3, 4$. The second F string is placed at some position $x_i = \bar{x}_i$ in the transverse four dimensional space.

After the S,T dualities discussed in section 2 we get the D1-D5 geometry in the R sector. The helical F string gives rise to the $AdS_3 \times S^3$ metric (3.11). The other F string is being regarded as a test string, so it does not contribute to the metric. After the dualities this F string becomes a D5 brane that wraps T^4, and thus appears as a 'string' in the remaining 5+1 dimensions. This worldsheet of this 'string' in these 5+1 dimensions can be parametrized as

$$x_i = \bar{x}_i, \quad y = R\sigma^1, \quad t = \sigma^0 \tag{4.6}$$

After the change of coordinates (2.8), (3.6) the embedding of this 'string' is[10] (we write $\chi = y/R$)

$$\theta = \bar{\theta}, \quad r = \bar{r}, \quad \chi = \sigma^1, \quad t = \sigma^0, \quad \tilde{\phi} = 0, \quad \tilde{\psi} = 0. \tag{4.7}$$

The string is still stationary, and so does not look yet look like a giant graviton. But after the coordinate change (2.10) bringing us to the NS sector we get the metric (3.11) of $AdS_3 \times S^3$, and the 'string' is described by

$$\theta = \bar{\theta}, \quad r = \bar{r}, \quad \chi = \sigma^1, \quad \psi = -\sigma^1, \quad \phi = -\sigma^0, \quad t = \sigma^0 \tag{4.8}$$

We see that we get the giant graviton profile (4.4), (4.5).[11]

4.2 Geometry produced by a giant graviton.

In the previous subsection we treated the giant graviton as a probe string on the undeformed $AdS_3 \times S^3$ background. This is a traditional view of a giant graviton, but

[9]Since we have two F strings with different winding numbers, we have drawn the strings in Fig 3(a,b) as multiwound strings on $0 < y < 2\pi R'$ rather than the 'opened up' strings in Figs 1,2.

[10]In general the conditions (4.6) lead to $\tilde{\phi} = \bar{\tilde{\phi}}$, $\tilde{\psi} = \bar{\tilde{\psi}}$, but we can always shift angular coordinates to make $\bar{\tilde{\phi}} = \bar{\tilde{\psi}} = 0$.

[11]Note that we have obtained the configuration with the negative choice of sign in (4.4), (4.5). We cannot obtain the configuration with the other sign if we start from a spacetime which goes over to flat space at infinity, since the part of the metric which joined the $AdS_3 \times S^3$ region to flat infinity breaks the symmetry which reverses the angles on S^3. But after we take the limit (2.5) we do have such a symmetry, and so the conclusions that we reach with one choice of sign are the same as for the configurations with the other sign.

the approximation breaks down as the charge of the giant graviton becomes comparable with the charge of the original $AdS_3 \times S^3$. In the FP representation this means that the length of the "short" string which became the giant graviton becomes comparable with the length of the helical string which produced $AdS_3 \times S^3$. At this point the giant graviton cannot be considered as a test particle and one has to take into account its backreaction on the geometry[12].

We can have various distributions of giant gravitons and each such distribution has a different metric. To illustrate how all such metrics may be constructed we take a particular distribution of giant gravitons and construct the corresponding metric. Based on the discussion of the above subsection, we start with the following profile of a F string 'broken' into two parts:

$$\mathbf{F}(v) = a\mathbf{e}_1 \cos\left(\frac{2\pi v}{\xi l}\right) + a\mathbf{e}_2 \sin\left(\frac{2\pi v}{\xi l}\right), \quad 0 \le v < l\xi \tag{4.9}$$

$$\mathbf{F}(v) = b\mathbf{e}_1, \quad l\xi \le v < l \tag{4.10}$$

The first line of this equation gives a helical string which produces $AdS_3 \times S^3$, and the second line gives rise to a bound state of several giant gravitons made from the D5-brane; the location of these giant gravitons is given as in (4.6).

As before the relations (2.3) give the coefficient functions in the metric. After the change of coordinates (3.6) we get the following functions for the chiral null model:

$$H^{-1} = 1 + \frac{Q\xi}{r^2 + a^2\cos^2\theta} + \frac{Q(1-\xi)}{r^2 + a^2\sin^2\theta - 2b\sqrt{r^2+a^2}\sin\theta\cos\tilde{\phi} + b^2}, \tag{4.11}$$

$$A_\phi = -\frac{a^2 Q}{r^2 + a^2\cos^2\theta}\frac{2\pi}{l}\sin^2\theta, \quad K = \frac{Qa^2}{r^2 + a^2\cos^2\theta}\frac{1}{\xi}\left(\frac{2\pi}{l}\right)^2 \tag{4.12}$$

Substituting these functions in the metric of D1–D5 system (2.1) and going to the NS sector using (2.10) we find the geometry which is produced by a giant graviton whose location can be expressed parametrically in terms of world sheet variables as[13]

$$\theta = \arcsin\frac{b}{a}, \quad r = 0, \quad \chi = \sigma^1, \quad \psi = -\sigma^1, \quad \phi = -\sigma^0, \quad t = R\sigma^0. \tag{4.13}$$

The resulting metric is quite complicated and we will not write it down explicitly. Let us instead do the same procedure we did in order to get the Aichelburg–Sexl type solution (3.15): there instead of looking at a shock wave produced by a single particle on the big diameter of the sphere (whose angular coordinate was given by $\phi = \phi_0 + t/R$) we considered a set of such particles uniformly distributed over the diameter. This change implies that we perform an average over location ϕ_0 in the harmonic functions. Let us now follow a similar path for the giant gravitons. Instead of the profile (4.9) we consider

[12] The similar problem for giant gravitons on $AdS_5 \times S^5$ was addressed in [12].

[13] We have assumed $b < a$. For $b > a$ we get $\sin\theta = 1, r \neq 0$ and we get a giant graviton expanding on the AdS rather than on the sphere.

a profile where the second part of the F string is further broken up into a large number of identical segments, and the location of these segments is distributed uniformly on a circle of radius b in the $x_1 - x_2$ plane. For convenience we just write this profile as

$$\mathbf{F}(v) = a\mathbf{e}_1 \cos\left(\frac{2\pi v}{\xi l}\right) + a\mathbf{e}_2 \sin\left(\frac{2\pi v}{\xi l}\right), \quad 0 \le v < l\xi \tag{4.14}$$

$$\mathbf{F}(v) = b(\cos\phi_0 \mathbf{e}_1 + \sin\phi_0 \mathbf{e}_2), \quad l\xi \le v < l \tag{4.15}$$

where it is assumed that an average will be taken over ϕ_0 in the harmonic functions in (2.3). We then get

$$H^{-1} = 1 + \frac{Q\xi}{f_0} + \frac{Q(1-\xi)}{f_1}, \quad A_\phi = -\frac{a^2 Q}{f_0}\frac{2\pi}{l}\sin^2\theta, \quad K = \frac{Qa^2}{f_0}\frac{1}{\xi}\left(\frac{2\pi}{l}\right)^2 \tag{4.16}$$

where

$$f_0 = r^2 + a^2\cos^2\theta, \quad f_1 = \left[(r^2 + a^2\sin^2\theta - b^2)^2 + 4r^2b^2\cos^2\theta\right]^{1/2} \tag{4.17}$$

Let us note that unlike the Aichelburg–Sexl type solution which had constant dilaton in the near horizon limit, the solution (4.16) has a non-constant dilaton even in the near horizon region:

$$e^{2\Phi} = \frac{a^2}{\xi}\left(\frac{2\pi}{l}\right)^2\left[\xi + (1-\xi)\frac{f_0}{f_1}\right]^{-1} \tag{4.18}$$

Looking at the harmonic functions (4.16) near infinity we find the D1 and D5 charges of the solution:

$$Q_1 = \frac{Qa^2}{\xi}\left(\frac{2\pi}{l}\right)^2, \quad Q_5 = Q \tag{4.19}$$

For later use it is convenient to rewrite the expression for A_ϕ from (4.16) in terms of the charge Q_1:

$$A_\phi = -\frac{\xi Q_1}{f_0}\frac{l}{2\pi}\sin^2\theta = -\frac{\xi Q_1}{f_0}\frac{n_5}{R}\sin^2\theta \tag{4.20}$$

where in the last step we used the relation between the length l of the F string appearing in the harmonic functions and the radius R of the y circle obtained after dualizing to D1-D5:

$$l = \frac{2\pi n_5}{R} \tag{4.21}$$

It is convenient to introduce the perturbation parameter $q \equiv 1 - \xi$ instead of ξ; thus $q = 0$ gives the $AdS_3 \times S^3$ spacetime without any giant gravitons. Then in the near horizon limit we obtain the following solution[14]:

$$ds_E^2 = e^{\Phi}\left\{Q_5(1-q)\left[-\frac{r^2+a^2}{Q_1Q_5}dt^2 + \frac{r^2}{Q_1Q_5}dy^2 + \frac{dr^2}{r^2+a^2}\right]\right.$$

[14]Here and below by ds_E^2 we denote the six dimensional metric in the Einstein frame

$$
\begin{aligned}
&+ \quad Q_5(1-q)\left[d\theta^2 + \sin^2\theta(d\tilde{\phi} - \frac{dt}{R})^2 + \cos^2\theta(d\tilde{\psi} - \frac{dy}{R})^2\right] \qquad (4.22)\\
&+ \quad \frac{qf_0}{Q_1}(-dt^2+dy^2) + \frac{qQ_5 f_0}{f_1}\left[d\theta^2 + \frac{dr^2}{r^2+a^2}\right]\\
&+ \quad \frac{qQ_5}{f_1}\left[r^2\cos^2\theta d\tilde{\psi}^2 + (r^2+a^2)\sin^2\theta d\tilde{\phi}^2\right]\Bigg\} \qquad (4.23)
\end{aligned}
$$

$$
\begin{aligned}
B^{RR}_{t\tilde{\psi}} &= -\frac{Q_5}{R}(1-q)\cos^2\theta, \quad B^{RR}_{y\tilde{\phi}} = -\frac{Q_5}{R}(1-q)\sin^2\theta, \quad B^{RR}_{ty} = \frac{f_0}{Q_1}\\
B^{RR}_{\tilde{\phi}\tilde{\psi}} &= \frac{Q_5(1-q)}{2}\cos 2\theta + \frac{Q_5 q(r^2\cos 2\theta + b^2 - a^2\sin^2\theta)}{2f_1} \qquad (4.24)
\end{aligned}
$$

The above solution was constructed in the Ramond sector. We can now perform the spectral flow to the NS sector using the relation (2.10). Then we get the metric

$$
\begin{aligned}
ds^2_E &= e^{\Phi}\Bigg\{Q_5(1-q)\left[-\frac{r^2+a^2}{Q_1Q_5}dt^2 + \frac{r^2dy^2}{Q_1Q_5} + \frac{dr^2}{r^2+a^2} + d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2\right]\\
&+ \quad \frac{qf_0}{Q_1}(-dt^2+dy^2) + \frac{qQ_5f_0}{f_1}\left[d\theta^2 + \frac{dr^2}{r^2+a^2}\right]\\
&+ \quad \frac{qQ_5}{f_1}\left[r^2\cos^2\theta(d\psi + \frac{dy}{R})^2 + (r^2+a^2)\sin^2\theta(d\phi + \frac{dt}{R})^2\right]\Bigg\} \qquad (4.25)
\end{aligned}
$$

which describes the metric created by the above discussed distribution of giant gravitons in $AdS_3 \times S^3$.

In the above we have separated part of the F charge away from the FP bound state to create the giant gravitons, and this mapped after S,T dualities to giant gravitons arising from D5 branes wrapped on the T^4. We can also start by separating off momentum modes (P) from the FP bound state, which implies that some of the momentum exists as separate massless particles moving along the y direction instead of being the momentum of traveling waves on the F string. The dualities map these momentum modes to giant gravitons created by D1 branes in $ADS_3 \times S^3$. We can also consider giant gravitons that are bound states of D1 and D5 branes – these arise from breaking the FP string into two parts each of which have nonzero F and P charges. For small F',P' charges in the separated part we can still regard the corresponding giant graviton as pointlike, but note that the momentum P' gives the F' string a finite transverse size, which translates to a nonzero transverse size for the D1-D5 giant graviton. This is of course the same phenomenon discussed in [5] which gives all D1-D5 bound states a nonzero size, and has been the essential physics giving rise to the different metrics (2.1) for the D1-D5 bound state.

5 Asymptotic behavior of the supergravity solution and its relation to dispersions in the CFT dual

The general solution (2.1) gives metrics for all R ground states, or equivalently, all chiral primaries if we take the limit (2.5) and perform spectral flow to the NS sector. In this section we examine these metrics in the limit (2.5), and observe the leading corrections to the spacetime $AdS_3 \times S^3$. We will see that this analysis gives us a way to infer certain characteristics of the CFT state from the asymptotic behavior of the corresponding geometry.

The simplest geometries are the special class (3.12); these metrics are locally exactly $AdS_3 \times S^3$. Thus at infinity in AdS space we find only a gauge field which is pure gauge, and scalars like the dilaton Φ are fixed at their background value. The corresponding CFT states (3.13) are particularly simple; all the twist operators σ_n have the same order and the same spin $(--)$. By contrast the Aichelburg-Sexl type solution (3.15) is *not* locally $AdS_3 \times S^3$, so if we examine the metric near infinity we will find corrections that fall off with various powers of r. The corresponding CFT state (3.27) has twist operators of different orders, so that we find a *dispersion* in the values of the n_i.

We start with a more detailed analysis of the asymptotic behavior of the Aichelburg-Sexl type solution (3.15), and then study the asymptotic behavior of the general metrics (2.1) in the limit (2.5). We observe from this analysis that there is a close relation between the existence of corrections at infinity and the presence of dispersions in the orders and spins of the $\sigma_n^{\epsilon\epsilon'}$ in the CFT state.

5.1 Asymptotic behavior of The Aichelburg-Sexl type solution (3.15)

The leading correction at $r \to \infty$ to the metric on S^3 is

$$h_{ab} = \frac{qL^2}{r^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos^4\theta & 0 \\ 0 & 0 & -\sin^4\theta \end{pmatrix} \tag{5.1}$$

(We use indices $a, b \ldots$ for S^3 coordinates and $\mu, \nu \ldots$ for the AdS_3.) We wish to put this perturbation into the basis which diagonalizes the quadratic part of the supergravity Lagrangian. Such a diagonalization was performed for the case $AdS_5 \times S^5$ in [13] and for the case $AdS_3 \times S^3$ in [14]. We write

$$h = h^a_a, \quad h_{(ab)} = h_{ab} - \frac{1}{3} g_{ab} h, \quad \nabla^b h_{(ab)} = 0 \tag{5.2}$$

where all indices are raised and lowered with the background metric $AdS_3 \times S^3$, and the last equation indicates the choice of de-Donder gauge. To bring the perturbation (5.1) to the de-Donder gauge we make the diffeomorphism

$$x^a \to x^a + \xi^a, \quad \xi_\theta = \frac{1}{4}\frac{qL^2}{r^2}\sin 2\theta, \quad \xi_\phi = \xi_\psi = 0 \tag{5.3}$$

We then get

$$\tilde{h}_{ab} = h_{ab} + \nabla_a \xi_b + \nabla_b \xi_a = (\frac{qL^2}{r^2}\cos 2\theta) g_{ab} \tag{5.4}$$

Thus the traceless part of the perturbation vanishes ($\tilde{h}_{(ab)} = 0$), and

$$\tilde{h} = \frac{3qL^2}{r^2}\cos 2\theta \tag{5.5}$$

so that $\tilde{h}$ is in the second harmonic on S^3.

Now we consider the perturbation of B^{RR}_{ab}. This perturbation is

$$b = \frac{qL^2}{r^2}\sin^2\theta\cos^2\theta d\psi \wedge d\phi \tag{5.6}$$

The diffeomorphism (5.3) shifts this to

$$\tilde{b} = \frac{2qL^2}{r^2}\sin^2\theta\cos^2\theta d\psi \wedge d\phi \tag{5.7}$$

We note that this field already satisfies the Lorentz gauge condition $\nabla^a b_{ab} = 0$ so we do not need to perform a further gauge transformation to bring it to the form used in [14].

We now note that the perturbations h, b mix with each other in the quadratic supergravity action, and the values (5.5), (5.7) are precisely those[15] that give the scalar σ in the second harmonic. This scalar has [14]

$$m_\sigma^2 = k(k-2), \quad k \geq 1 \tag{5.8}$$

Since we have $k = 2$ we get $m^2 = 0$. From (5.5) we see that the amplitude of this scalar falls off at infinity as $1/r^2$, which is the correct falloff for a massless scalar in AdS_3.

5.2 Asymptotic behavior of the general metrics (2.1)

Now consider the general solution (2.1). We assume the limit (2.5) has been taken so that for $r \to \infty$ we are looking near infinity of AdS space rather than infinity of flat space.

[15]The normalization for the 2-form potential in [14] is such that the action is $-\frac{1}{3}H^2$. We use instead the normalization that is conventional for the 10-dimensional IIB theory where the action is $-\frac{1}{12}H^2$.

5.2.1 Dilaton and volume of T^4

First we note that the scalars Φ (the dilaton) and V (the volume of T_4) are both functions of $H(1+K)$. Thus we expand $H(1+K)$ for large r to see if these scalars are excited. We will use Cartesian coordinates for a part of the following analysis, but note that at leading order the radial coordinate of AdS_3 space is

$$r^2 = x_i x_i \tag{5.9}$$

and the angular variables describe the S^3.

In the expressions (2.3) we expand

$$|x-F|^{-2} = \frac{1}{x^2}[1 + \frac{2x\cdot F}{x^2} - \frac{F^2}{x^2} + \frac{4(x\cdot F)^2}{x^4} + \ldots] \tag{5.10}$$

where the dots imply higher order terms in $1/r$. We define for any function of v the average

$$\frac{1}{l}\int_0^l S(v)dv \equiv \langle S\rangle \tag{5.11}$$

We then find

$$H(1+K) = \langle(\dot{F})^2\rangle + \frac{2x_i}{x^2}[\langle(\dot{F})^2 F_i\rangle - \langle F_i\rangle\langle(\dot{F})^2\rangle] + \ldots \tag{5.12}$$

Note that if

$$(\dot{F})^2 = constant \tag{5.13}$$

then the correction to $H(1+K)$ vanishes at leading order. We thus see that the above scalars are excited at order $1/r$ at infinity if and only if we have a *dispersion* in the quantity $(\dot{F})^2$. Recall that the functions $F_i(v)$ give, after Fourier transformation, the orders and spins of the twist operators $\sigma_n^{\epsilon\epsilon'}$ creating the dual CFT state (eqn. (2.11)). Thus the dispersion of $(\dot{F})^2$ is directly related to the dispersion of the orders and spins of these twist operators. The metric (3.12) had the profile $F_i(v)$ in the form of a uniform helix, and thus had $(\dot{F})^2 = constant$. The Aichelburg-Sexl solution had a profile $F_i(v)$ which was composed of helical as well as straight line segments, which imply non–constant $(\dot{F})^2$. But the limits (ii') in section 3.3 imply an 'effectively constant' value for $(\dot{F})^2$ as far as its use in (5.12) is concerned, and the dispersion in (5.12) vanishes. This is in accord with the fact that the above scalars are not excited in the Aichelburg-Sexl solution.

5.2.2 The metric perturbation

We now address the corrections to the metric. Note that the 6-dimensional Einstein metric is obtained (upto an overall constant) from the 10-dimensional string metric (2.1) by simply dropping the term $dz_\alpha dz_\alpha$; this follows because the 6-dimensional dilaton is constant for these solutions. We rewrite the resulting metric in a form that gives the dimensional reduction to AdS_3, since it is this form of the metric that will give the excitations dual to operators in the CFT. We get

$$ds^2 = \mu_{ij}(dx_i + C_i dt + D_i dy)(dx_j + C_j dt + D_j dy) + \nu_1 dt^2 + \nu_2 dy^2 + \sigma dt dy \tag{5.14}$$

with

$$\begin{aligned}
\mu_{ij} &= \sqrt{\frac{1+K}{H}}\delta_{ij} - \sqrt{\frac{H}{1+K}}(A_i A_j - B_i B_j) \\
C_i &= \mu_{ik}^{-1}\sqrt{\frac{H}{1+K}}A_k, \quad D_i = \mu_{ik}^{-1}\sqrt{\frac{H}{1+K}}B_k \\
\nu_1 &= -\sqrt{\frac{H}{1+K}} - \mu_{ij}C_i C_j, \quad \nu_2 = \sqrt{\frac{H}{1+K}} - \mu_{ij}D_i D_j \\
\sigma &= -\frac{1}{2}\mu_{ij}C_i D_j
\end{aligned} \tag{5.15}$$

We will later convert the x_i to r and angular coordinates on S^3; the coordinates r, t, y will form the AdS space.

5.2.3 The correction $\sigma dt dy$

The special metrics (3.12), (3.15) do not have any term mixing dt and dy, so we would like to understand more generally what conditions imply a vanishing of σ in the metric (5.14). Note that the vanishing of $dtdy$ mixing in the R sector implies the vanishing of $dtd\chi$ mixing in the metric obtained for the NS sector after dimensional reduction to the t, r, χ spacetime. To leading order μ_{ij} is proportional to δ_{ij}, so

$$\sigma = 0 \;\Rightarrow\; A_i B_i = 0 \tag{5.16}$$

Expanding near infinity we find

$$A_i = -Q[\frac{1}{x^2}\langle\dot{F}_i\rangle + \langle\frac{2x\cdot F}{x^4}\dot{F}_i\rangle] = -2Q\langle\dot{F}_i F_j\rangle\frac{x_j}{x^4} \tag{5.17}$$

where we have used the fact that $F_i(v+L) = F_i(v)$, which implies that for any quantity $S(v)$ made from the F_i

$$\langle\dot{S}\rangle = 0 \tag{5.18}$$

We define

$$a_{ij} = \langle\dot{F}_i F_j\rangle \tag{5.19}$$

Using (5.18) we find

$$a_{ij} = -a_{ji} \tag{5.20}$$

Thus we get at leading order

$$A_i = -2Qa_{ij}\frac{x_j}{x^4} \tag{5.21}$$

From (2.4) we find

$$B_i = -2Qb_{ij}\frac{x_j}{x^4}, \quad b_{ij} = \frac{1}{2}\epsilon_{ijkl}a_{kl} \tag{5.22}$$

Since a_{ij} is an antisymmetric matrix we can perform an $SO(4)$ rotation on the coordinates x_i to bring it to the form

$$a_{12} = -a_{21} = \alpha, \quad a_{34} = -a_{43} = \beta \tag{5.23}$$

with all other components zero. The condition (5.16) then becomes, using (5.22)

$$a_{12}a_{34} = 0 \ \Rightarrow \ a_{12} = 0 \text{ or } a_{34} = 0 \tag{5.24}$$

If $a_{34} = 0$ then we find that the profile $F_i(v)$ describes vibrations only in the $x_1 - x_2$ plane; thus the condition for vanishing of σ is that there be no dispersion in the *orientation plane* of the string in the FP description of the metric. Rotation in a given plane corresponds by eqn. (2.11) to the spins $(\epsilon\epsilon')$ of all twist operators being in the same wavefunction, so in the CFT state we will have no dispersion in these spins.[16]

5.2.4 Corrections to the metric on S^3

Consider the corrections to

$$\mu_{ij} = \sqrt{\frac{K}{H}}\delta_{ij} - \sqrt{\frac{H}{K}}(A_iA_j - B_iB_j) \tag{5.25}$$

We will consider separately the corrections to the two terms in μ_{ij}, and combine the effects at the end. We find

$$\begin{aligned}\sqrt{\frac{1+K}{H}}dx_idx_i = \frac{Q}{x^2}\langle(\dot{F})^2\rangle^{1/2} \ &+ \ \frac{Q}{x^2}\langle(\dot{F})^2\rangle^{-1/2}\frac{x_i}{x^2}[\langle(\dot{F})^2F_i\rangle + \langle(\dot{F})^2\rangle\langle F_i\rangle] \\ &+\frac{Q}{x^2}\langle(\dot{F})^2\rangle^{1/2}\frac{x_ix_j}{x^4}T_{ij} + \ldots \end{aligned} \tag{5.26}$$

with

$$\begin{aligned} T_{ij} \ &= \ \frac{1}{2}\langle(1+\frac{(\dot{F})^2}{\langle(\dot{F})^2\rangle})(-F^2\delta_{ij} + 4F_iF_j)\rangle \\ &\quad - \frac{1}{2}[\langle F_i\rangle - \langle\frac{(\dot{F})^2}{\langle(\dot{F})^2\rangle}F_i\rangle]\ [\langle F_j\rangle - \langle\frac{(\dot{F})^2}{\langle(\dot{F})^2\rangle}F_j\rangle] \end{aligned} \tag{5.27}$$

[16] We have decided to do the spectral flow from the R to the NS sector using a specific choice of $U(1) \times U(1)$ out of the $\mathcal{R}$-symmetry group $SU(2) \times SU(2)$; this choice was made in eqn. (2.8), (2.10). The rotation that brings a_{ij} to the form (5.23) maps the set x_1, x_2, x_3, x_4 to a new orthonormal set $\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4$. Thus in eqn. (5.24) the vanishing of a_{34} is the vanishing of a in the $\hat{x}_3\hat{x}_4$ plane, though we did not indicate this in the equation to avoid cumbersome notation. A string rotating in the $x_1 - x_2$ plane would imply spins $(-\,-)$ for all twist operators, but a string rotating along $\hat{x}_1 - \hat{x}_2$ implies that all twist operators have some wavefunction (ϵ, ϵ') obtained by rotating the spin $(-\,-)$ by the appropriate $SO(4) = SU(2) \times SU(2)$ group element.

The term of order $1/r^3$ in (5.26) can be canceled by a shift

$$x_i = x'_i + v_i, \quad v_i = \frac{1}{2}[\langle \frac{(\dot{F})^2}{\langle(\dot{F})^2\rangle} F_i\rangle + \langle F_i\rangle] \tag{5.28}$$

(We will henceforth drop the primes on the shifted coordinates to avoid cumbersome notation.) This shift adds to T_{ij} a contribution

$$\tilde{T}_{ij} = [v^2\delta_{ij} - 4v_i v_j] \tag{5.29}$$

We see that we get a simplification in the form of T_{ij} if $(\dot{F})^2$ is constant; this is the same condition (5.13) that led to the vanishing of dilaton and torus volume scalars. In this case we get

$$T_{ij}^{total} = T_{ij} + \tilde{T}_{ij} = 4[\langle F_i F_j\rangle - \langle F_i\rangle\langle F_j\rangle] - \delta_{ij}[\langle F_k F_k\rangle - \langle F_k\rangle\langle F_k\rangle] \tag{5.30}$$

We see that the above expression is also written as a sum of dispersions, but these dispersions cannot all vanish in any configuration since that would imply that the F_i be constants.

5.3 Asymptotic fields for constant $(\dot{F})^2$

We now write down expressions for the asymptotic form of some metric components, for the case where there is no dispersion in $(\dot{F})^2$.

5.3.1 The metric on S^3

We have analyzed above the first contribution to μ_{ij} in (5.25), and we now compute the contribution of the gauge fields A_i, B_i. We assume that the matrix a_{ij} has been put in the form (5.23). We map the x_i to polar coordinates using (2.8). We find

$$\begin{aligned} ds^2_{gauge} &\equiv -\sqrt{\frac{H}{K}}(A_i A_j - B_i B_j) dx_i dx_i \\ &= \frac{Q}{|\dot{F}|r^2}\left[-(\alpha\sin^2\theta d\phi + \beta\cos^2\theta d\psi)^2 + (\beta\sin^2\theta d\phi - \alpha\cos^2\theta d\psi)^2\right] \end{aligned} \tag{5.31}$$

From (5.17), (5.18) we see that the nonvanishing of the A_i, B_i at infinity can be itself regarded as arising from a nonzero value for the dispersion

$$\langle \dot{F}_i F_j\rangle - \langle \dot{F}_i\rangle\langle F_j\rangle = \langle \dot{F}_i F_j\rangle \tag{5.32}$$

To put the metric in de-Donder gauge we make a diffeomorphism similar to (5.3)

$$\xi_\theta = Q\frac{\alpha^2 - \beta^2}{4|\dot{F}|r^2}\sin 2\theta, \quad \xi_\phi = \xi_\psi = 0 \tag{5.33}$$

which yields

$$ds^2_{gauge} = Q\frac{\alpha^2-\beta^2}{|\dot F|r^2}\left[\cos 2\theta(d\theta^2+\cos^2\theta d\psi^2+\sin^2\theta d\phi^2)-\frac{4\alpha\beta}{\alpha^2-\beta^2}\sin^2\theta\cos^2\theta d\phi d\psi\right]$$

Combining the contributions from (5.26) and ds^2_{gauge} we get for the metric on S^3

$$\begin{aligned} ds^2_{sphere} &= Q|\dot F|d\Omega^2+\left[\frac{Q|\dot F|}{r^2}\left[4n_in_j\langle F_iF_j\rangle-\langle F^2\rangle\right]+\frac{Q(\alpha^2-\beta^2)}{|\dot F|r^2}\cos 2\theta\right]d\Omega^2 \\ &- \frac{4Q\alpha\beta}{|\dot F|r^2}\sin^2\theta\cos^2\theta d\phi d\psi \end{aligned} \tag{5.34}$$

where $n_i=\frac{x_i}{x}$.

We find that just as in the case of the Aichelburg–Sexl metric we get a contribution to trace h in the second harmonic, which is dual to an operator with $\Delta=2$ in the CFT.[17]

5.3.2 Gauge fields on AdS_3

Now we look at the asymptotic behavior of terms in the metric that mix the S^3 coordinates with the AdS_3 coordinates. This part of the metric turns out to be

$$ds_3^{mixed}=\frac{2}{|\dot F|}\left[(\alpha\sin^2\theta d\phi+\beta\cos^2\theta d\psi)dt+(\beta\sin^2\theta d\phi-\alpha\cos^2\theta d\psi)d\chi\right] \tag{5.35}$$

We note that since h^ϕ_μ, h^ψ_μ do not depend upon sphere coordinates and $h^\theta_\mu=0$ (here μ is an AdS index), the metric ds^2_3 is automatically in the gauge

$$\nabla_a h^a_\mu=\partial_a h^a_\mu-\Gamma^b_{ab}h^b_\mu=0 \tag{5.36}$$

We further find

$$\nabla^2 h_{a\mu}=-2h_{a\mu}, \tag{5.37}$$

(where the Laplacian is taken over the sphere). Comparing with the fields studied in [14] we see that such an excitation is in the harmonic $k=1$. The field $h_{a\mu}$ mixes with $B^{RR}_{a\mu}$, and the lightest field arising from this combination has scaling dimension $\Delta=1$ in the CFT.

[17]In general we can also find a contribution to h in the zeroth harmonic on S^3, with this contribution falling like $1/r^2$ at infinity. This power law suggests a massless scalar, but the quadratic supergravity action does not have such a scalar made from h. But the theory has cubic couplings, for example $h\partial\phi\partial\phi$ where ϕ is the dilaton. We can have $\phi\sim 1/r$, and also $\partial\phi\sim 1/r$ where the derivative is taken along the S^3. We then get from the cubic coupling a contribution $h\sim 1/r^2$, with h in the zeroth harmonic. Thus we have to be careful about higher order terms in the Lagrangian in the analysis of asymptotic fields.

6 Discussion

Let us summarize our main conclusions. In most treatments of AdS/CFT duality we start with the geometry $AdS_m \times S^n$ which is dual to the vacuum of the CFT. We then consider excitations over this vacuum perturbatively, for example in the computation of multipoint correlation functions of chiral primaries [15, 16]. For the case $AdS_3 \times S^3$ however we are able to write down exact supergravity solutions that describe arbitrary chiral primary states. These geometries are *not* small perturbations to $AdS_3 \times S^3$, though they go to $AdS_3 \times S^3$ at large r. The availability of these solutions allows us to probe questions that we cannot address at present using other $AdS_m \times S^n$ spaces.

At first it appears to be clear what a chiral primary is: we have one or more massless quanta of supergravity rotating around a diameter of S^3. But a further look indicates that the problem is more complex: we can find more than one family of metrics that have $\Delta = J$. Further, the work on giant gravitons suggests that the rotating quanta may themselves be finite size objects rather than pointlike particles, and this would create more complicated metrics after the backreaction is taken into account.

We found that two special families of metrics with $\Delta = J$ – the conical defect metrics and the Aichelburg-Sexl type solution – were special cases of the general set of chiral primaries that can be written down using the map to the FP system developed in [5]. But the CFT states dual to these two families were characterized by rather different distributions of twist operators – in the first case all twist operators were identical while in the second case there was a wide dispersion in their orders n_i.

The giant graviton solutions turned out to not describe states of the bound D1-D5 system at all – the bound states map under S,T dualities to a single F string carrying vibrations, while the giant gravitons correspond to breaking this string into two or more pieces. It is noteworthy that a giant graviton type solution can be written down not only for the space $AdS_3 \times S^3$ but also for all the geometries (2.1) which go over at infinity to flat space. This is easy to see: in the R sector we have to break off some of the D1 and D5 branes and separate them from the rest of the D1-D5 bound state by a transverse displacement. This solution always exists classically since for vanishing moduli there is no force between the D1-D5 bound state and transversely displaced D1 or D5 branes. The coordinate change (2.10) to the NS sector then makes this static D1 brane rotate, and we get a D1 brane that maintains its finite size while satisfying its classical equation of motion.

It is important to note that such a giant graviton, while appearing to be a supersymmetric solution at classical order, is not a supersymmetric solution in the exact theory. To see this consider for simplicity two parallel D-p branes, each wrapped on a finite torus T^p of volume V and placed a distance a apart. Classically this configuration has a mass $M \sim \frac{V}{g}$ where g is the string coupling, and a charge that is related to the mass by a BPS condition. But in order for the two branes to be localized at their respective positions we must make a superposition of their momentum eigenfunctions, and this adds in a 'localization energy' of order $\frac{p^2}{2M} \sim \frac{g}{a^2 V}$. Thus to get mass equaling charge we must either place the two branes in their zero momentum eigenfunctions – in which case their mean separation becomes infinity and we do not get a set of branes at finite separation a – or we take their (unique) bound state, where the wavefunction describing their relative separation is a very particular wavefunction and the exact solution is known to be BPS.

The above discussion extends to the geometry that we can create by putting N branes at each of the locations $x = \pm\frac{a}{2}$. Classically this geometry will appear supersymmetric, but for any finite N it is clear that corrections of order g will have to show us a breakdown of supersymmetry. (In a worldsheet description this breakdown should occur due to one loop open string diagrams.) Similarly we can conclude that the giant graviton solutions do not represent exactly supersymmetric states and are thus not chiral primaries.[18] We note that the case of $AdS_3 \times S^3$ is rather special however, and this conclusion may not carry over to giant gravitons in other $AdS_m \times S^n$ spaces.

Returning to the D1-D5 bound state solutions, we recall that the prescription of [15] allowed us to compute expectation values of operators in the NS vacuum of a CFT by using the dual supergravity. It appears plausible that given the solutions representing chiral primary states, there would be a way to compute expectation values of operators in such states, by a perturbative supergravity computation around these more complicated backgrounds. Even without setting up such a computational scheme, it appears reasonable that the behavior of the supergravity solutions near infinity contains information about the corresponding CFT state: for any supergravity mode the solution growing at $r \to \infty$ describes operators inserted at the boundary while the solution decaying at infinity describes the 'state' of the CFT [17].

In view of these expectations we have examined the $r \to \infty$ behavior of the general supergravity solution describing a chiral primary. We noted that the conical defect solutions were locally $AdS_3 \times S^3 \times T^4$, so there were no excitations at infinity apart from a pure gauge term mixing the S^3 with the AdS_3. The corresponding CFT state was created by a set of twist operators that were all identical. The Aichelburg-Sexl type solution did have nontrivial excitations of field at infinity, and there was a dispersion in the orders of the twist operators. We found that this was somewhat of a general pattern – we could relate the strength of several fields at infinity to some *dispersion* in the twist operators creating the state. It would be interesting to see if such a principle holds more generally in the AdS/CFT correspondence.

[18] We have taken the F string in the FP solution to be described by a classical vibration profile. This is a coherent state excitation and thus not an energy eigenstate, but in principle we can consider the energy eigenstates instead and then we would have exact chiral primary states. If we try to make energy eigenstates out of the giant graviton solutions then the giant gravitons delocalize all the way to infinity in AdS_3 since the potential is flat; such a delocalization does not happen for the bound states which are described by a single F string carrying momentum P.

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A A chiral null model approach to D1-D5 solutions

In this appendix we will recall the construction of [9] which related solutions for the vibrating string with solutions describing the D1–D5 system.

We begin with a fundamental string which carries momentum. The general solution for such a string is given by a chiral null model [18]:

$$\begin{aligned} ds^2 &= H'(\vec{x}', v')\left(-du'\, dv' + K'(\vec{x}', v')dv'^2 + 2A'_i(\vec{x}', v')dx'_i dv'\right) + d\vec{x}' \cdot d\vec{x}' + d\vec{z}' d\vec{z}', \\ B_{uv} &= -G_{uv} = \frac{1}{2}H'(\vec{x}', v'), \qquad B_{vi} = -G_{vi} = -H'(\vec{x}', v')A'_i(\vec{x}', v'), \\ & \qquad e^{-2\Phi} = H'^{-1}(\vec{x}', v'). \end{aligned} \tag{A.1}$$

Regarding A'_i as a gauge field we can construct the field strength $\mathcal{F}_{ij} = A'_{j,i} - A'_{i,j}$. The functions in the chiral null model are required satisfy the equations

$$\partial^2 H'^{-1} = 0, \qquad \partial^2 K' = 0, \qquad \partial_i \mathcal{F}^{ij} = 0. \tag{A.2}$$

Here ∂^2 is the Laplacian in the x'_i coordinates. Note that the indices i, j span the subspace $\{x_i\}$ where the metric is just δ_{ij}, and thus these indices are raised and lowered by this flat metric.

We can now perform the chain of dualities which relates the vibrating string with D1–D5 system:

$$\left|\begin{array}{c}\mathrm{P}\\ \\ \mathrm{F1}\end{array}\right| \xrightarrow{S} \left|\begin{array}{c}\mathrm{P}\\ \\ \mathrm{D1}\end{array}\right| \xrightarrow{T6789} \left|\begin{array}{c}\mathrm{P}\\ \\ \mathrm{D5}\end{array}\right| \xrightarrow{S} \left|\begin{array}{c}\mathrm{P}\\ \\ \mathrm{NS5}\end{array}\right| \xrightarrow{T56} \left|\begin{array}{c}\mathrm{F1}\\ \\ \mathrm{NS5}\end{array}\right| \xrightarrow{S} \left|\begin{array}{c}\mathrm{D1}\\ \\ \mathrm{D5}\end{array}\right| \tag{A.3}$$

Note that before performing the S duality relating (P,D5) and (P,NS5) we also perform an electric–magnetic duality (see [9] for details). The resulting D1–D5 geometry is:

$$\begin{aligned} ds^2 &= \sqrt{\frac{H}{1+K}}\left[-(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2\right] + \sqrt{\frac{1+K}{H}} d\vec{x} \cdot d\vec{x} \\ &+ \sqrt{H(1+K)} d\vec{z} \cdot d\vec{z} \end{aligned} \tag{A.4}$$

$$\begin{aligned} e^{2\Phi} &= H(1+K), \qquad C^{(2)}_{ti} = \frac{B_i}{1+K}, \qquad C^{(2)}_{ty} = -\frac{K}{1+K}, \\ C^{(2)}_{iy} &= -\frac{A_i}{1+K}, \qquad C^{(2)}_{ij} = C_{ij} + \frac{A_i B_j - A_j B_i}{1+K} \end{aligned} \tag{A.5}$$

The functions H, K and A_i appearing in this solution have the same values as H', K' and A'_i:

$$H(\vec{x}) = H'(\vec{x}'), \qquad K(\vec{x}) = K'(\vec{x}'), \qquad A_i(\vec{x}) = A'_i(\vec{x}'), \tag{A.6}$$

and the forms B_i and C_{ij} are defined by

$$dC = -^*dH^{-1}, \qquad dB = -^*dA. \tag{A.7}$$

Here Hodge dual is taken with respect to the four dimensional space x^1, x^2, x^3, x^4 with flat metric. Note that due to the equations of motion for the null chiral model (A.2):

$$d^*dH^{-1} = 0, \qquad d^*dA = 0 \tag{A.8}$$

the equations (2.4) can be integrated to give the forms C and B.

If we consider the metric created by a single elementary string, then the parameters of the chiral null model are not arbitrary, but they are determined in terms of the vibration profile $\mathbf{F}(v)$:

$$H^{-1} = 1 + \frac{Q}{l}\int_0^l \frac{dv}{(\mathbf{x}-\mathbf{F})^2}, \quad K = \frac{Q}{l}\int_0^l \frac{|\dot{F}|^2 dv}{(\mathbf{x}-\mathbf{F})^2}, \quad A_i = -\frac{Q}{l}\int_0^l \frac{\dot{F}_i dv}{(\mathbf{x}-\mathbf{F})^2} \tag{A.9}$$

As an example we consider the helical profile:

$$F_1 = a\cos\frac{2\pi m v}{l}, \quad F_2 = a\sin\frac{2\pi m v}{l}, \quad F_3 = F_4 = 0. \tag{A.10}$$

For this profile it is convenient to make a change of coordinates (2.8), (3.6). Then in terms of the new coordinates $r, \theta, \tilde{\phi}, \tilde{\psi}$ we find the geometry of the D1–D5 system [8]:

$$\begin{aligned} ds^2 &= -\frac{f_0}{\sqrt{f_1 f_5}}(dt^2 - dy^2) + \sqrt{f_1 f_5}\left(\frac{dr^2}{r^2+a^2} + d\theta^2\right) + \sqrt{\frac{f_1}{f_5}}\sum_{i=1}^{4} dz^i dz^i \\ &+ \frac{\sqrt{f_1 f_5}}{f_0}\left[(r^2 + \frac{a^2 Q_1 Q_5 \cos^2\theta}{f_1 f_5})\cos^2\theta d\tilde{\psi}^2 + (r^2 + a^2 - \frac{a^2 Q_1 Q_5 \sin^2\theta}{f_1 f_5})\sin^2\theta d\tilde{\phi}^2\right] \\ &- \frac{2 Q_1 Q_5 a}{\sqrt{f_1 f_5}}\left[\sin^2\theta dt d\tilde{\phi} + \cos^2\theta dy d\tilde{\psi}\right] \end{aligned} \tag{A.11}$$

$$\begin{aligned} e^{2\tilde{\phi}} &= \frac{f_1}{f_5}, \qquad C^{(2)}_{ty} = -\frac{2Q_1}{f_1}, \qquad C^{(2)}_{t\tilde{\psi}} = -\frac{\sqrt{Q_1 Q_5} a \cos^2\theta}{f_1}, \\ C^{(2)}_{y\tilde{\phi}} &= -\frac{\sqrt{Q_1 Q_5} a \sin^2\theta}{f_1}, \qquad C^{(2)}_{\tilde{\phi}\tilde{\psi}} = Q_5\cos^2\theta + \frac{Q_5 a^2 \sin^2\theta\cos^2\theta}{f_1}. \end{aligned} \tag{A.12}$$

Here we have introduced three convenient functions:

$$f_0 = r^2 + a^2\cos^2\theta, \qquad f_1 = f_0 + Q_1, \qquad f_5 = f_0 + Q_5. \tag{A.13}$$

and the charges are

$$Q_5 = Q, \quad Q_1 = Q\left(\frac{2\pi a m}{l}\right)^2 \tag{A.14}$$

In the near horizon limit ($r \ll (Q_1 Q_5)^{1/4}$) the metric reduces to (3.12) with $\gamma = \frac{1}{m}$.

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